GENERATING THE MAPPING CLASS GROUPS BY TORSIONS

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ABSTRACT. Let S_g be the closed oriented surface of genus g and let $\mathrm{Mod}(S_g)$ be the mapping class group. When the genus is at least 3, $\mathrm{Mod}(S_g)$ can be generated by torsion elements. We prove the follow results. For $g \geq 4$, $\mathrm{Mod}(S_g)$ can be generated by 4 torsion elements. Three generators are involutions and the forth one is an order 3 element. $\mathrm{Mod}(S_3)$ can be generated by 5 torsion elements. Four generators are involutions and the fifth one is an order 3 element.

1. INTRODUCTION

Let S_g be the closed oriented surface of genus g. The mapping class group $\mathrm{Mod}(S_g)$ is defined by $\mathrm{Homeo}^+(S_g)/\mathrm{Homeo}_0(S_g)$, the group of homotopy classes of oriented-preserving homeomorphisms of S_g.

The study of the generating set of the mapping class group of an oriented closed surface begun in the 1930s. The first generating set of the mapping class group was found by Dehn ([2]). This generating set consist of $2g(g-1)$ Dehn twists for genus $g \geq 3$. About a quarter of a century later, Lickorish found a generating set consisting of $3g - 1$ Dehn twists for $g \geq 1$ ([8]). The numbers of Dehn twists in the generating set was improved to $2g + 1$ by Humphries ([4]). In fact this is the minimal number of the generators if we require that all of the generators are Dehn twists. This is also proved by Humphries.

If we consider generators other than Dehn twists, it is possible to find smaller generating sets. Wajnryb found that the mapping class groups $\mathrm{Mod}(S_g)$ can be generated by two elements ([12]), each of which is some product of Dehn twists. One element in Wajnrb's generating set has finite order. Korkmaz in [7] gave another generating set consisting of two elements. One of Korkmaz's generators is a Dehn twist and the other one is a torsion element of order $4g + 2$.

The mapping class group $\mathrm{Mod}(S_g)$ can be generated only by torsion elements. By an *involution* in a group we simply mean any element of order 2. Luo proved that for $g \geq 3$, $\mathrm{Mod}(S_g)$ is generated by finitely many involutions ([9]). In Luo's result, the number of the involutions grows linearly with g. Brendle and Farb found a universal upper bound for the number of involutions that generate $\mathrm{Mod}(S_g)$. They proved that for $g \geq 3$, $\mathrm{Mod}(S_g)$ can be generated by 7 involutions ([1]). Kassabov improved the number of involution generators to 4 if $g \geq 7$, 5 if $g \geq 5$ and 6 if $g \geq 3$ ([6]).

For the extended mapping class group, Stukow proved that $\mathrm{Mod}^{\pm}(S_g)$ can be generated by three orientation reversing elements of order 2 ([11]).

If we consider torsion elements of higher orders, not only involutions, Brendle and Farb in [1] also found a generating set consisting of 3 torsion elements. The minimal generating set of torsion elements was found by Korkmaz. He showed that $\mathrm{Mod}(S_g)$ can be generated by two torsion elements, each of which is of order $4g + 2$ ([7]). For generating set of smaller order, Monden found that for $g \geq 3$, $\mathrm{Mod}(S_g)$ is generated by 3 elements of order 3, or generated by 4 elements of order 4 ([10]).

Since the group generated by two involution is a dihedral group, Farb and Margalit in [3], Kassabov in [6] and Monden in [10] mentioned that the following problem is open:

Problem 1.1. *Whether or not $Mod(S_g)$ can be generated by three involutions?*

In this paper, we give generating sets consisting of 3 involutions and a torsion of order 3:

Theorem 1.2. *For $g \geq 4$, $Mod(S_g)$ can be generated by 4 torsion elements, among which 3 elements are involutions and the forth one is an order 3 torsion. For $g = 3$, $Mod(S_3)$ can be generated by 5 torsion elements, among which 4 elements are involutions and the fifth one is an order 3 torsion.*

Remark 1.3. *When $g < 7$, the number of involutions in Kassabov's result is at least 5. We give a smaller generating set in these cases. For $g \geq 4$ cases, our result uses the generating sets of the same form.*

Remark 1.4. *Monden's result use only 3 torsions of order 3. Korkmaz's result use only 2 torsions of order $4g + 2$. Our advantage is the use of involutions.*

2. PRELIMINARIES

We use the convention of functional notation, namely, elements of the mapping class group are applied right to left. The composition fh means that h is applied first.

Let c be the isotopy class of a simple closed curve on S_g. Then the (left-hand) *Dehn twist* T_c about c is the homotopy class of the homeomorphism obtained by cutting S_g along c, twisting one of the sides by 2π to the left and gluing two sides of c back to each other. See Fig. 1.
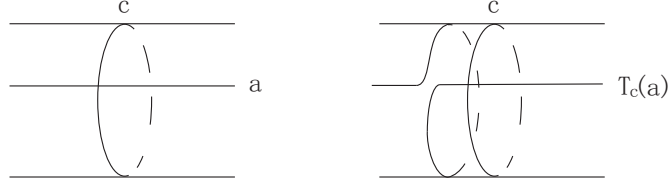

Fig. 1

We recall the following results (see, for instance, [3]):

Lemma 2.1 (Conjugacy relation). *For any $f \in Mod(S_g)$ and any isotopy class c of simple closed curves in S_g, we have:*

$$T_{f(c)} = f \, T_c \, f^{-1}.$$

Lemma 2.2 (Dehn twists along disjoint curves commute). *Let a, b be two simple closed curves on S_g. If a is disjoint from b, then*

$$T_a T_b = T_b T_a.$$

Lemma 2.3 (Braid relation). *Let a, b be two simple closed curves on S_g. If the geometric intersection number of a and b is one, then*

$$T_a T_b T_a = T_b T_a T_b.$$

Let $c_1, c_2, \ldots, c_t$ be simple closed curves on S_g. If $i(c_j, c_{j+1}) = 1$ for all $1 \le j \le k-1$ and $i(c_j, c_k) = 0$ for $|j - k| > 1$, then we call $c_1, c_2, \ldots, c_t$ is a *chain*. Take a regular neighbourhood of their union. The boundary of this neighbourhood consist of one or two simple closed curves, depending on whether t is even or odd. In the odd case, denote the boundary curve by d. In the even case, denote the boundary curves by d_1 and d_2. We have the following *chainrelation*:

Lemma 2.4 (Chain relation).
$$
\begin{aligned}
(T_{c_1} T_{c_2} \ldots T_{c_t})^{2t+2} &= T_d & t \text{ even,} \\
(T_{c_1} T_{c_2} \ldots T_{c_t})^{t+1} &= T_{d_1} T_{d_2} & t \text{ odd.}
\end{aligned}
$$

Dehn discovered the lantern relation and Johnson rediscovered it ([5]). This is a very useful relation in the theory of maping class groups.

Lemma 2.5 (Lantern relation). *Let a, b, c, d, x, y, z be the curves showed in Figure 2 on a genus zero surface with four boundaries. Then*
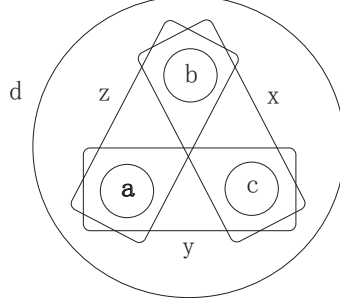
$$T_a T_b T_c T_d = T_x T_y T_z$$

Fig. 2

Lickorish's result on the set of generators is the following:

Lemma 2.6 (Lickorish's generators)**.** *The mapping class group $Mod(S_g)$ is generated by the set of Dehn twists $\{T_{a_i}\ (1 \leq i \leq g),\ T_{b_i}\ (1 \leq i \leq g),\ T_{a_i}\ (1 \leq i \leq g-1)\}$ (See Fig. 3).*
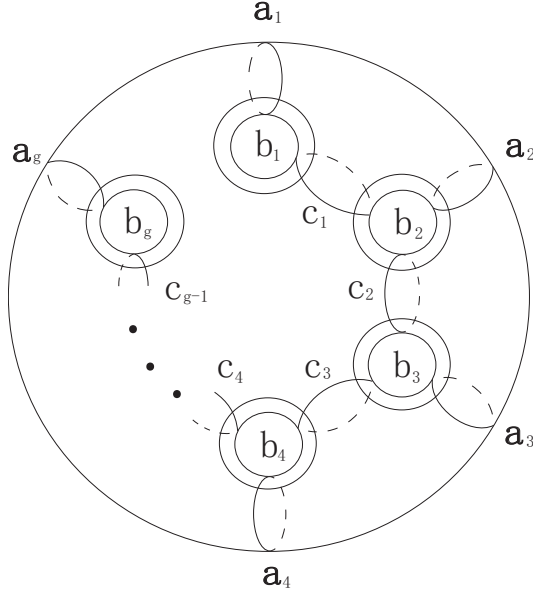

Fig. 3

3. MAIN RESULTS

Now we are ready to prove the main results of this paper.

Theorem 3.1. *For $g \geq 4$, $Mod(S_g)$ can be generated by 4 torsion elements, among which 3 elements are involutions and the forth one is an order 3*

torsion. For $g = 3$, $\mathrm{Mod}(S_3)$ can be generated by 5 torsion elements, among which 4 elements are involutions and the fifth one is an order 3 torsion.

Proof. According to Lickorish's theorem, we only need to construct a set of torsion elements such that each Dehn twist of the Lickorish curves $\{T_{a_i} \, (1 \leq i \leq g), \, T_{b_i} \, (1 \leq i \leq g), \, T_{c_i} \, (1 \leq i \leq g-1)\}$ can be generated by these torsion generators. By the conjugacy relation of Dehn twists, if the group G generated by some torsions has the following two properties, then it is $\mathrm{Mod}(S_g)$:

property 1, under the action of G, $a_i{}'s, b_i{}'s, c_i{}'s$ are in the same orbit;

property 2, some T_{a_i} is in G.

There are involutions f_1, f_2 on the surface such that f_1 and f_2 are π-rotations and $f_2 f_1$ is an order g element, hence $f_2 f_1$ permutes $a_i{}'s$, $b_i{}'s$, $c_i{}'s$ respectively (See Fig. 3).
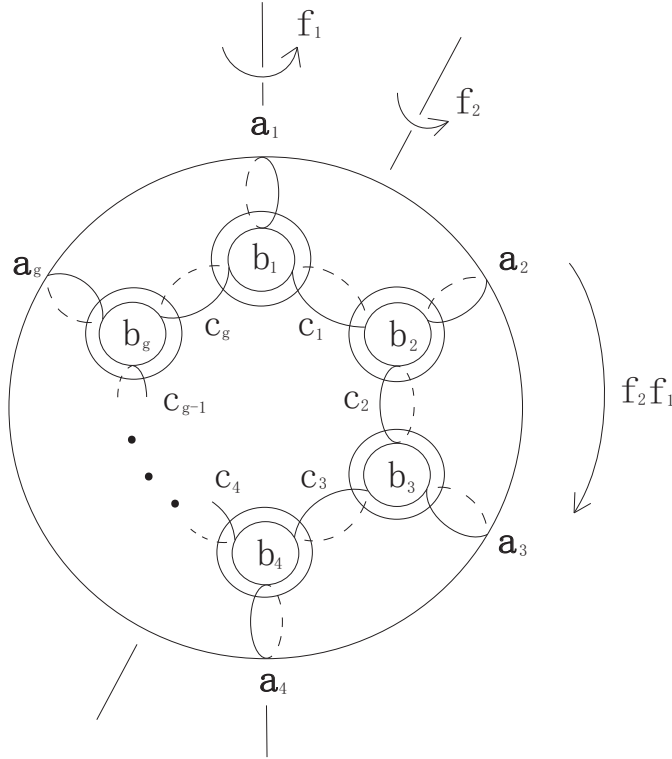

Fig. 4

Meanwhile, in the lantern relation (see the previous figure 2), since each curve in $\{a, b, c, d\}$ is disjoint from x, y, z, and each curve in $\{a, b, c, d\}$ is disjoint from each other, the Dehn twistss along them are commutative. The lantern relation can be rewritten as

$$T_d = (T_x T_a^{-1})(T_y T_b^{-1})(T_z T_c^{-1}).$$

Locally, there is a $(2\pi/3)$-rotation h on the genus zero surface with four boundary, mapping the curve d to itself, sending a to b, b to c, c to a, x to y, y to z and z to x. Hence

$$T_d = (T_x T_a^{-1})(h T_x T_a^{-1} h^{-1})(h^2 T_x T_a^{-1} h^{-2}).$$

Compare figure 4 to figure 2, there is a genus-zero-and-four-boundary subsurface bounded by $\{a_1,\ c_1,\ c_2,\ a_3\}$. a_2 separates $\{a_1, c_1\}$ from $\{c_2, a_3\}$ on this subsurface. The position of $\{a_1, c_1, a_2\}$ on the subsurface in figure 4 is like the position of $\{a, d, x\}$ in figure 2. T_d and $T_x T_a^{-1}$ in figure 2 is like T_{c_1} and $T_{a_2} T_{a_1}^{-1}$. If we can extend h to a global order 3 homeomorphism f_3 on S_g, then

$$T_{c_1} = (T_{a_2} T_{a_1}^{-1})(f_3 T_{a_2} T_{a_1}^{-1} f_3^{-1})(f_3^2 T_{a_2} T_{a_1}^{-1} f_3^{-2}).$$

Now we decompose $T_{a_2} T_{a_1}^{-1}$ into the product of two involutions f_2 and $T_{a_1} f_2 T_{a_1}^{-1}$:

$$T_{a_2} T_{a_1}^{-1} = (f_2 T_{a_1} f_2) T_{a_1}^{-1} = f_2(T_{a_1} f_2 T_{a_1}^{-1}).$$

Notice that $f_2 f_1$ send c_1 to c_2 and f_3 permutes c_2, a_1 and a_3. Hence the subgroup $G \leq \mathrm{Mod}(S_g)$ generated by $\{f_1, f_2, T_{a_1} f_2 T_{a_1}^{-1}, f_3\}$ includes all the T_{a_i}'s and T_{c_i}'s. It remains to show how we get T_{b_i}'s by torsion elements. We only need to construct a torsion sending some a_i to b_i.

In the case of genus $g \geq 4$, we make the global order 3 homeomorphism f_3 sending some a_i to b_i as follows. See figure 5.
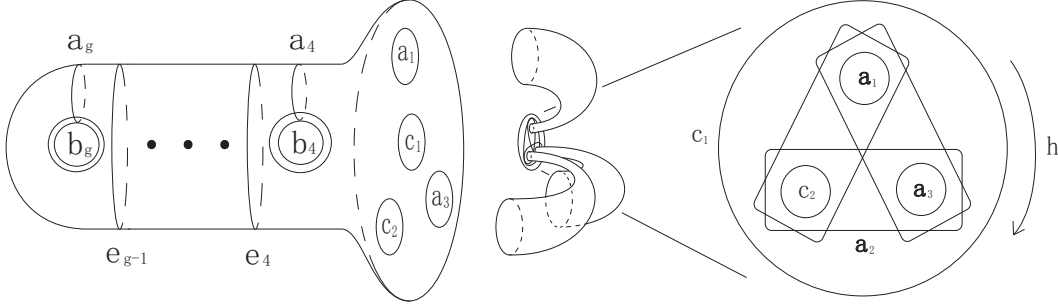

Fig. 5

h is an order 3 local homeomorphism on the subsurface with four boundaries. Each point on this subsurface (including the point on the boundary) has period 3 except the fix point at the center. The complement of the such a subsurface is a genus $g-3$ subsurface with also four boundaries a_1, a_3, c_1, c_2.

On the complement, we need to construct a periodic map with the following properties: (1) c_1 is mapped to itself; (2) a_1, a_3, c_2 are permuted cyclically; (3) each point on the complement subsurface (including the point on the boundary) has period 3 except the fix points; (4) some a_i is sent to b_i.

We cut the complement along curves $e_4, \ldots, e_{g-1}$ into $g-3$ pieces, each of which is of genus 1. See figure 5. When $g \geq 5$, such pieces can be divided into 3 classes: class (i): has only one boundary e_{g-1}; class (ii): has 5 boundaries e_4, a_1, a_3, c_2, c_1; class (iii): has 2 boundaries e_{i-1}, e_i. When $g = 4$, the complement has only one piece with 4 boundaries a_1, a_3, c_2, c_1.

To construct the periodic maps of order 3 on such pieces, sending some meridian a_i to the longitude b_i, take the genus 1 surfaces as the quotient space of the hexagon (perhaps with holes) gluing the opposite sides. See figure 6. The $2\pi/3$-rotation of the plane is the obvious homeomorphism on the quotient space we want.

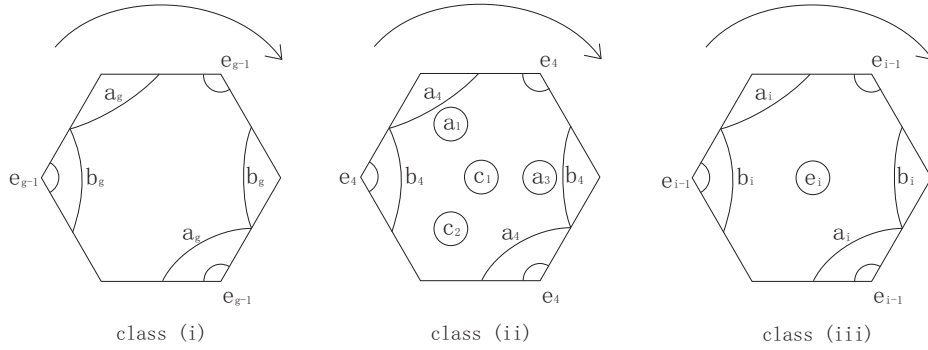class (i)   class (ii)   class (iii)

Fig. 6

Now at the beginning, we have the order 3 homeomorphism h on the lantern. Then we can extend the order 3 homeomorphism to the adjacent piece, inducing an order 3 homeomorphism on the rest component of the boundary of the adjacent piece. Then piece by piece, we make a global order 3 homeomorphism f_3 of the genus g surface. All the $a_i{}'s$, $b_i{}'s$, $c_i{}'s$ is in the same orbit under the action of the group G generated by $\{f_1, f_2, T_{a_1} f_2 T_{a_1}^{-1}, f_3\}$. T_{a_1} is in G. Hence $G = \mathrm{Mod}(S_g)$.

In the case of genus $g = 3$, the complement of the lantern bounded by a_1, a_3, c_2, c_1 is a surface of genus zero has four boundaries. The global order 3 homeomorphism f_3 on complement of the lantern can only be the same form as h. f_3 cannot send some a_i to some b_j. We need one more involution to send some a_i to some b_j. Now a_3 and b_3 are two non-separating curves on the surface. So there is a homeomorphism $\sigma = T_{a_3} T_{b_3}$ sending a_3 to b_3 and fixing a_1, b_1, a_2, b_2. Then $\sigma^{-1} f_1 \sigma$ is an involution, sending a_3 to b_2. The group generated by $\{f_1, f_2, T_{a_1} f_2 T_{a_1}^{-1}, f_3, \sigma^{-1} f_1 \sigma\}$ is $\mathrm{Mod}(S_3)$.

$\square$

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